Eur. J. Phys. 11 (1990) 60–62. Printed in the UK

Tidal braking of the Earth's rotation—a study in cubism

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Received 13 December 1988, 22 September 1989

Abstract. The braking of the Earth's rotation by solid-body tides raised by the Moon, and the concomitant recession of the Moon are estimated by a very elementary method.

Résumé. Une méthode très élémentaire est utilisée pour calculer le ralentissement de la rotation de la Terre dû aux marées lunaires agissant sur la matière solide de la planète, ainsi que l'éloignement de la Lune elle-même.

1. Introduction

In a recent paper, Sonnet *et al* (1988) extracted information from terrestrial sediments about the variation of the Moon's orbit over geological periods of time: they inferred that the Moon has been receding from the Earth over the last 7×10^8 years at an average rate of just over two centimetres per year.

This recession is customarily explained through the dissipation of the Earth's tidal energy: the tidal bulge raised on the Earth by the Moon has a small phase difference with respect to the instantaneous direction of the Moon, producing a decelerative torque on the Earth's rotation and an opposite torque transferring energy to the Moon, which therefore recedes.

Until recently, it has been assumed that tidal energy is dissipated in the solid Earth; it is now more usual to attribute a substantial part of the loss to frictional dissipation in shallow seas (Hansen 1982), although direct calculations of shallow-sea dissipation do not entirely account for the observed slowing down of the Earth's rotation.

The purpose of this paper is to present a very simple analysis of the braking of the Earth's rotation through tidal forcing of the entire planet; the analysis is based on the same approach that I used (Celnikier 1983) to highlight the essential physics which underlies the prediction of volcanic activity on Io, and shows that, just as in the case of Io, the terrestrial problem is basically a function of two parameters, neither of which is well known, but one of which is particularly uncertain.

2. A simple cubic model of the Earth

Consider as in Celnikier (1983) a homogeneous cubic body of side L, density ρ and mass $M = L^3\rho$. This will be taken as an adequate approximation to calculate the tidal forcing of the real planet by the Moon; it will in fact be convenient to think of it as two halves, each of mass $M/2$, whose barycentres are separated by $L/2$.

Suppose first that this cubic object were orientated permanently with one face perpendicular to the radius vector of the Moon. The Earth would then be distorted by a differential gravitational force ΔF acting on two points separated by $L/2$:

$$\Delta F \simeq \Delta(GMm/2R^2) = GMmL/2R^3 \qquad (1)$$

where m is the mass of the Moon and R its geocentric distance.

We can think of the cube as acted upon by two opposing forces $\Delta F/2$ with respect to the overall barycentre, so that the total work W done to stretch the cube is given by:

$$W = 2\frac{\Delta L}{2}\frac{\Delta F}{2} = \frac{\Delta L \Delta F}{2} = \frac{L\Delta F^2}{4SY}$$

if the material satisfies Hook's law of elasticity; Y is Young's modulus, and S is the surface area of one face.

Now, for most ordinary solid materals:

$$Y \simeq 3\mu$$

where μ is the bulk modulus of elasticity, so that using equation (1)

$$W = \frac{L}{12S\mu}\left(\frac{GMmL}{2R^3}\right)^2 = \frac{G^2M^2m^2}{48\mu R^6}L. \qquad (2)$$

3. Tidal braking of a rotating Earth

The Earth rotates with respect to the Earth–Moon

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line, and so a tidal deformation is created in one particular direction; different parts of the Earth pass through this region and so are successively 'stretched' and 'relaxed'. This situation can be modelled for our 'cubic' planet by thinking of each pair of faces as being stretched, and then released over one quarter of a rotation period: the solid body of the planet is in this sense set into a forced oscillation. During such a process, a fraction of the oscillation energy is dissipated; the inverse of this quantity is the 'Q-factor' or 'dissipation function'.

Therefore, at each complete rotation of the (cubic) Earth, an amount of energy E_t is lost to the mechanical motion:

$$E_t = 4W/Q \tag{3}$$

$$= \frac{G^2M^2m^2}{12\mu R^6 Q}L. \tag{4}$$

The Earth's rotational kinetic energy E_r must diminish to 'pay' for this dissipation:

$$E_r = 0.5I(\mathrm{d}\theta/\mathrm{d}t)^2$$

where $\mathrm{d}\theta/\mathrm{d}t$ is the planet's axial angular velocity and I its moment of inertia; for a cube of side L:

$$I = ML^2/6$$

so that:

$$E_r = \tfrac{1}{2}ML^2(\mathrm{d}\theta/\mathrm{d}t)^2.$$

E_t/E_r gives the fractional change of rotational energy, $\Delta E_r/E_r$ per planetary rotation; since:

$$\Delta E_r/E_r = 2\Delta P/P$$

where P is the rotation period, one has immediately:

$$\Delta P/P = \frac{G^2Mm^2}{2\mu QR^6L(\mathrm{d}\theta/\mathrm{d}t)^2} \text{ per day.} \tag{5}$$

It is interesting to note that this result is relatively insensitive to the shape of the planet. On the one hand, we can assess a rough correction to the tidal energy loss (equation (4)) by noting that it applies to the volume of the body; since the volume of a sphere of diameter L is roughly one half of the volume of a cube of side L, one would expect the tidal loss for a spherical planet to be about one half of that given by equation (4). On the other hand, the moment of inertia of a sphere of diameter L is also about one half of that of a cube of side L: consequently, the ratio E_t/E_r is hardly changed.

One can see immediately the essential difficulty in obtaining a theoretical estimate of tidal braking: μ and Q are not well known for bulk planetary material. This was discussed in some detail in Celnikier (1983) and the reader is referred to that paper for details and references; essentially, μ can be taken as $10^{11}\,\mathrm{N\,m^{-2}}$, with a probable uncertainty of one significant figure (it is amusing to note that this is just the 'universal' value one deduces *ab initio* using the simple argument of Weisskopf (1975)), while Q can be taken as 100 with an uncertainty of an order of magnitude either way. These values are consistent with the tidal activity of Io; whether this is relevant is not clear.

Substituting these numbers into equation (5), one obtains:

$$\Delta P/P \simeq 10^{-11} \text{ per year.}$$

Astrometric measurements spread over the last couple of centuries have yielded a fractional change in the length of the day of about 2×10^{-10} per year (Glass *et al* 1977); on the other hand, the paleontological evidence of Sonnett *et al* (1988) suggests an average fractional change over the last 7×10^8 years of about 10^{-10} per year.

We can obtain an expression for the time taken to change the period by a significant amount by writing equation (5) in the form:

$$\Delta P/P = \frac{G^2Mm^2}{2\mu QR^6L(2\pi/P)^2}\Delta t$$

and integrating:

$$\frac{1}{2P_1^2} - \frac{1}{2P_2^2} = \frac{G^2Mm^2}{2\mu QR^6L(2\pi)^2}\tau \tag{6}$$

where τ is the time taken (in units of present days) for the rotation period to increase from P_1 to P_2 (measured in seconds).

4. Lunar recession

The Earth is not only losing energy through tidal braking, but also angular momentum; the angular momentum of the Moon increases, and so it recedes. The process will terminate (if we ignore the influence of the Sun) when the period of the Earth's axial rotation is equal to the orbital period of the Moon.

This problem is usually handled via the torque exerted by the tidally defomed Earth on the Moon; this of course furnishes the mechanism of the recession, but we do not need to use it to assess the importance of the phenomenon.

The current angular momentum of the Moon, $J_{\mathrm{Moon,now}}$, is substantially due to its orbital motion:

$$J_{\mathrm{Moon,now}} = mR^2(\mathrm{d}\theta/\mathrm{d}t)^2_{\mathrm{Moon,now}} \simeq 3 \times 10^{34}.$$

The present angular momentum of the Earth, $J_{\mathrm{Earth,now}}$, is substantially that of its axial rotation:

$$J_{\mathrm{Earth,now}} \simeq I(\mathrm{d}\theta/\mathrm{d}t)^2_{\mathrm{Earth,now}} \simeq 1.6 \times 10^{34}.$$

Therefore, the total angular momentum of the Earth–Moon system, J_{tot}, is just the sum of these two components:

$$J_{\mathrm{tot}} \simeq 4.6 \times 10^{34}.$$

When the Earth's rotation is synchronised with the lunar orbit, its rotational angular momentum can be neglected, so that the Moon's orbital motion will furnish the dominant contribution to the total angular momentum:

$$J_{\text{Moon,final}} \simeq mR^2_{\text{final}}(\mathrm{d}\theta/\mathrm{d}t)^2_{\text{Moon,final}}.$$

Now, orbital equilibrium requires that:

$$GM/R^2 = v^2_{\text{Moon,final}}/R_{\text{final}} = R_{\text{final}}(\mathrm{d}\theta/\mathrm{d}t)^2_{\text{Moon,final}}$$

so that:

$$J_{\text{Moon,final}} \simeq mR^{1/2}_{\text{final}}(GM)^{1/2} = J_{\text{tot}}$$

since angular momentum should be conserved, whence:

$$R_{\text{final}} = (4.6 \times 10^{34})^2/GMm^2 \simeq 10^9 \text{ m}$$
$$\simeq 2.5 \times \text{current distance of Moon.}$$

The time scale for this recession can be estimated from equation (6) by putting P_1 equal to the present length of the day, and setting $1/P_2$ equal to zero, since under synchronous conditions the rotation period must be over an order of magnitude longer than it is now; this gives a timescale of 4×10^{10} years. Consequently, with the values of the constants μ and Q as chosen and neglecting all effects other than solid-body tides, the Moon would be receding at an average rate of something like 2 cm per year: the paleontological analysis of Sonnett *et al* (1988) leads to an average recession rate over the last 7×10^8 years of just over 2 cm per year, while the Apollo laser ranging experiments suggest a value of about 3.5 cm per year.

5. A few concluding remarks

The simple calculation presented in this paper has sidestepped many sources of difficulty: the lunar orbit is not circular, the Earth is not a cube and is not homogeneous, we have neglected the effect of the Sun, etc. However, when these effects are explicitly included, they unfortunately tend to mask the essential feature that the basic constants on which any analysis (simple or complicated) depends are very badly known and I have tried to show in this paper how to analyse the problem in a simple way which highlights the importance of the very badly known Q value of the material.

In spite of the elementary nature of the calculation, the results are within an order of magnitude of observation.

Rather more detailed calculations suggest that solid-body tides account better for ancient values of the lunar recession (as measured, for example, by the average over a long time scale) when oceanic effects are supposed to have been less, than for the current values (as measured by the Apollo experiment). This is part of the rationale behind the belief that oceanic tides are currently the major contributing factor to tidal braking; amusingly enough, our simple calculation seems to go in the same sense, but one should perhaps not give too much credence to factors of two in an estimate which intrinsically cannot be more accurate than an order of magnitude.

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